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Citation for final published version:

Dadarlat, Marius and Pennig, Ulrich ORCID: <https://orcid.org/0000-0001-5441-6130> 2017. Connective C*-algebras. Journal of Functional Analysis 272 (12) , pp. 4919-4943. 10.1016/j.jfa.2017.02.009 file

Publishers page: <http://dx.doi.org/10.1016/j.jfa.2017.02.009>
<<http://dx.doi.org/10.1016/j.jfa.2017.02.009>>

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CONNECTIVE C^* -ALGEBRAS

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ABSTRACT. Connectivity is a homotopy invariant property of separable C^* -algebras which has three notable consequences: absence of nontrivial projections, quasidiagonality and a more geometric realization of KK-theory for nuclear C^* -algebras using asymptotic morphisms. The purpose of this paper is to further explore the class of connective C^* -algebras. We give new characterizations of connectivity for exact and for nuclear separable C^* -algebras and show that an extension of connective separable nuclear C^* -algebras is connective. We establish connectivity or lack of connectivity for C^* -algebras associated to certain classes of groups: virtually abelian groups, linear connected nilpotent Lie groups and linear connected semisimple Lie groups.

1. INTRODUCTION

Connectivity of separable C^* -algebras was introduced in our earlier paper [10] under different terminology, see Definition 2.1 below. The initial motivation for studying it stemmed from our search for homotopy-symmetric C^* -algebras. By a result of Loring and the first author [9], these are precisely the separable C^* -algebras for which one can unsuspend in the E-theory of Connes and Higson [6]. Using a result of Thomsen [34], we proved in [10] that connectivity is equivalent to homotopy-symmetry for all separable nuclear C^* -algebras. Moreover, we showed that connectivity has a number of important permanence properties. These facts allowed us to exhibit new classes of homotopy-symmetric C^* -algebras.

The purpose of this paper is to further explore the class of connective C^* -algebras. We are motivated by the following three properties they share:

- (i) If A is a separable nuclear C^* -algebra, then $KK(A, B)$ is isomorphic to the homotopy classes of completely positive and contractive (cpc) asymptotic morphisms from A to $B \otimes \mathcal{K}$ for any separable C^* -algebra B .
- (ii) Connective C^* -algebras are quasidiagonal. In fact, if A is connective, then $A \otimes B$ is quasidiagonal for any C^* -algebra B . ($A \otimes B$ denotes the minimal tensor product.)

M.D. was partially supported by NSF grant #DMS-1362824.

(iii) Connective C^* -algebras do not have nonzero projections. In fact, if A is connective, then $A \otimes B$ does not have nonzero projections for any C^* -algebra B .

Connectivity is of particular interest in the case of group C^* -algebras. A countable discrete group G is called connective if the augmentation ideal $I(G)$ defined as the kernel of the trivial representation $\iota: C^*(G) \rightarrow \mathbb{C}$ is a connective C^* -algebra. In view of properties (ii) and (iii) connectivity of G may be viewed as a stringent topological property that accounts simultaneously for the quasidiagonality of $C^*(G)$ and the verification of the Kadison-Kaplansky conjecture for certain classes of groups. Examples of nonabelian connective groups were exhibited in [10] and [11].

In this paper we give new characterizations of connectivity for exact and nuclear separable C^* -algebras, see Prop. 2.2, 2.3. We prove that connectivity of separable nuclear C^* -algebras is preserved under extensions, see Thm. 2.4. This is a key permanence property which does not hold for quasidiagonal C^* -algebras.

There is a close connection between the topology of the spectrum and connectivity, which we employ to reveal an obstruction to connectivity by using work of Blackadar and Cuntz [1] and Pasnicu and Rørdam [30]. In particular, we show that a countable discrete group G is not connective if the trivial representation ι is a shielded point of the unitary dual of G in the sense of Def. 2.9.

Motivated by this, we give a complete description of the neighborhood of ι in the spectrum of the Hantzsche-Wendt group G , which is a torsion free crystallographic group with holonomy $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, in Sec. 3.1. This allows us to prove that G is not connective in this case (Cor. 3.2). Moreover, we show that this group provides a counterexample to a conjecture from [8]. Specifically, we prove that the natural map $[[I(G), \mathcal{K}]] \rightarrow K^0(I(G))$ is not an isomorphism (Lem. 3.3). In contrast, we show that all torsion free crystallographic groups with *cyclic holonomy* are connective (Thm. 3.8).

Next, we investigate connectivity for C^* -algebras associated to Lie groups. We show that all noncompact linear connected nilpotent Lie groups have connective C^* -algebras (Thm. 4.3). Using classic results from representation theory in conjunction with permanence properties of connectivity, we show that if G is a linear connected complex semisimple Lie group, then $C_r^*(G)$ is connective if and only if G is not compact (Thm. 4.5). Moreover, if G is a linear connected real reductive Lie group, then $C_r^*(G)$ is connective if and only if G does not have a compact Cartan subgroup (Thm. 4.6).

A common denominator of our results concerning group C^* -algebras is that in all the cases we analyzed, $C^*(G)$ contains a large connective ideal.

2. CONNECTIVE C^* -ALGEBRAS

2.1. Definitions and background. For a C^* -algebra A , the *cone over A* is defined as $CA = C_0[0, 1) \otimes A$ the *suspension of A* as $SA = C_0(0, 1) \otimes A$.

The first of the following two notions was introduced in [10, Def. 2.6 (i)], under a different terminology which we have abandoned. We use the abbreviation *cpc map* for a completely positive and contractive map.

Definition 2.1. Let A be a C^* -algebra.

(a) A is *connective* if there is a $*$ -monomorphism

$$\Phi: A \rightarrow \prod_n CL(\mathcal{H}) / \bigoplus_n CL(\mathcal{H})$$

which is liftable to a cpc map $\varphi: A \rightarrow \prod_n CL(\mathcal{H})$.

(b) A is *almost connective*, if there is a (not necessarily liftable) $*$ -monomorphism $\Phi: A \rightarrow \prod_n CL(\mathcal{H}) / \bigoplus_n CL(\mathcal{H})$.

For a discrete group G , we define $I(G)$ to be the augmentation ideal, i.e. the kernel of the trivial representation $C^*(G) \rightarrow \mathbb{C}$. We will sometimes say that a discrete amenable group G is connective if the C^* -algebra $I(G)$ is connective. Note that (almost) connective C^* -algebras do not have nonzero projections. Thus any connective C^* -algebra is nonunital. Our definition allows that the zero C^* -algebra $\{0\}$ is connective.

Let A and B be C^* -algebras. An *asymptotic morphism* is a family of maps $\{\varphi_t: A \rightarrow B\}_{t \in [0, \infty)}$ such that

- a) for each $a \in A$ the map $t \mapsto \varphi_t(a)$ is norm-continuous and bounded,
- b) for all $a, b \in A$ and $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \|\varphi_t(a + \lambda b) - (\varphi_t(a) + \lambda \varphi_t(b))\| &= 0 \\ \lim_{t \rightarrow \infty} \|\varphi_t(ab) - \varphi_t(a) \varphi_t(b)\| &= 0 \\ \lim_{t \rightarrow \infty} \|\varphi_t(a^*) - \varphi_t(a)^*\| &= 0. \end{aligned}$$

A *discrete asymptotic morphism* $(\varphi_n)_{n \in \mathbb{N}}$ between A and B is a family of maps $\varphi_n: A \rightarrow B$ that satisfies the analogous conditions as a) and b) above with the index set replaced by \mathbb{N} . A homotopy between two (discrete) asymptotic morphisms $(\varphi_t^0)_{t \in I}$ and $(\varphi_t^1)_{t \in I}$ is a (discrete) asymptotic morphism $H_t: A \rightarrow C[0, 1] \otimes B$, such that $\text{ev}_i \circ H_t = \varphi_t^i$ for all $t \in I$, where I either denotes $[0, \infty)$ or \mathbb{N} . We will say that a (discrete) asymptotic morphism $(\varphi_t)_{t \in I}$ is completely positive and contractive (cpc) if each of the maps φ_t is cpc. The corresponding homotopy classes will be denoted as follows:

- $[[A, B]]$ – homotopy classes of asymptotic morphisms,
- $[[A, B]]_{\mathbb{N}}$ – homotopy classes of discrete asymptotic morphisms,
- $[[A, B]]_{\mathbb{N}}^{\text{cpc}}$ – homotopy classes of discrete cpc asymp. morphisms

2.2. Characterizations of connectivity. In the following we give two more characterizations of connectivity for exact and respectively nuclear C^* -algebras.

Proposition 2.2. *Let A be a separable exact C^* -algebra. Then A is connective if and only if there is an injective $*$ -homomorphism $\pi: A \rightarrow \mathcal{O}_2$ which is null-homotopic as a discrete cpc asymptotic morphism. This means that $[[\pi]] = 0$ in the set $[[A, \mathcal{O}_2]]_{\mathbb{N}}^{cp}$.*

Proof. (\Rightarrow) By assumption, there is a cpc discrete asymptotic morphism $\{\varphi_n: A \rightarrow C[0, 1] \otimes \mathcal{O}_2\}_n$ such that $\varphi_n^{(0)} = \pi$ is an injective $*$ -homomorphism and $\varphi_n^{(1)} = 0$. Thus, we can view $\{\varphi_n\}_n$ as an injective discrete asymptotic morphism $\{\varphi_n: A \rightarrow C_0[0, 1] \otimes \mathcal{O}_2 \subset CL(\mathcal{H})\}_n$ and hence A is connective.

(\Leftarrow) Suppose that A is a separable exact connective C^* -algebra. By [10, Prop. 2.11] it follows that $[[A, \mathcal{O}_2 \otimes \mathcal{K}]]_{\mathbb{N}}^{cp}$ is an abelian group. By Kirchberg's embedding theorem, there is an injective $*$ -homomorphism $\pi: A \rightarrow \mathcal{O}_2 \otimes \mathcal{K}$. Moreover $\pi \oplus \pi: A \rightarrow \mathcal{O}_2 \otimes \mathcal{K}$ is unitarily homotopy equivalent to π . It follows that $[[\pi]] \oplus [[\pi]] = [[\pi]]$ in the group $[[A, \mathcal{O}_2 \otimes \mathcal{K}]]_{\mathbb{N}}^{cp}$ and hence $[[\pi]] = 0$. After embedding $\mathcal{O}_2 \otimes \mathcal{K}$ into \mathcal{O}_2 we obtain the desired conclusion. \square

Proposition 2.3. *Let A be a separable nuclear C^* -algebra. The following properties are equivalent.*

- (i) A is connective.
- (ii) $A \otimes B$ is connective for some C^* -algebra B that contains a nonzero projection.
- (iii) $A \otimes B$ is connective for all C^* -algebras B
- (iv) $[[A, \mathcal{O}_2 \otimes \mathcal{K}]] = 0$.
- (v) $[[A, L(\mathcal{H}) \otimes \mathcal{K}]] = 0$.

Proof. The equivalences (i) \Leftrightarrow (ii) \Leftrightarrow (iii) were established in [10]. (For (ii) \Rightarrow (i) observe that A is a subalgebra of $A \otimes B$ if B contains a nonzero projection.)

(i) \Rightarrow (iv) and (i) \Rightarrow (v). Let B be a σ -unital C^* -algebra such that $KK(A, B) = 0$, for instance $B = \mathcal{O}_2$ or $B = L(\mathcal{H})$. If A is connective, then A is homotopy symmetric and hence $[[A, B \otimes \mathcal{K}]] \cong KK(A, B) = 0$ by [10, Thm. 3.1]. Note that even though [10, Thm. 3.1] was stated for separable C^* -algebras B it is routine to extend the result to general C^* -algebras using the separability of A .

(iv) \Rightarrow (i) and (v) \Rightarrow (i). Fix an embedding $\pi: A \rightarrow \mathcal{O}_2 \subset L(\mathcal{H}) \otimes \mathcal{K}$ and regard it as a constant asymptotic morphism $\{\pi_t: A \rightarrow L(\mathcal{H}) \otimes \mathcal{K}\}_t$. By assumption, $[[\pi_t]] = 0$ in $[[A, L(\mathcal{H}) \otimes \mathcal{K}]]$ and hence by restriction, $[[\pi_n]] = 0$ in $[[A, L(\mathcal{H}) \otimes \mathcal{K}]]_{\mathbb{N}}$. We shall view $L(\mathcal{H}) \otimes \mathcal{K}$ as a subalgebra of $L(\mathcal{H})$. The corresponding homotopy from the constant discrete morphism $\{\pi_n\}_n$ to zero

will induce an embedding $\Phi: A \rightarrow \prod_n CL(\mathcal{H}) / \bigoplus_n CL(\mathcal{H})$ which is liftable to a cpc map $\varphi: A \rightarrow \prod_n CL(\mathcal{H})$ by the nuclearity of A . \square

2.3. Extensions of connective C^* -algebras. Connectivity of C^* -algebras has a plethora of permanence properties as proven in [10, Thm. 3.3]. In particular, it is inherited by split extensions [10, Thm. 3.3 (d)]. In the following theorem this result is extended to non-split extensions as well.

Theorem 2.4. *Let $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$ be an exact sequence of separable nuclear C^* -algebras. If J and B are connective, then A is connective.*

Proof. Since connectivity passes to nuclear subalgebras we may replace the given extension by

$$0 \rightarrow J \otimes \mathcal{O}_2 \otimes \mathcal{K} \rightarrow A \otimes \mathcal{O}_2 \otimes \mathcal{K} \rightarrow B \otimes \mathcal{O}_2 \otimes \mathcal{K} \rightarrow 0.$$

Adding to this extension a trivial absorbing extension, using the addition in Ext-theory, we obtain an absorbing extension

$$(1) \quad 0 \rightarrow J \otimes \mathcal{O}_2 \otimes \mathcal{K} \rightarrow E \rightarrow B \otimes \mathcal{O}_2 \otimes \mathcal{K} \rightarrow 0,$$

which by construction has the property that $A \subset A \otimes \mathcal{O}_2 \otimes \mathcal{K} \subset E$, see for instance [4, Lemma 2.2]. Since $\text{Ext}(B \otimes \mathcal{O}_2, J \otimes \mathcal{O}_2) = 0$ as \mathcal{O}_2 is KK-contractible and we are dealing with an absorbing extension, it follows that the extension (1) splits by [24, Sec. 7] and so E is connective by [10, Thm. 3.3]. We conclude that $A \subset E$ is connective. \square

In the sequel we will need to use the following result from [10], which is based on [2].

Theorem 2.5 ([10], Cor. 3.4.). *Let A be a separable continuous field of nuclear C^* -algebras over a compact connected metrizable space X . If one of the fibers of A is connective, then A is connective.*

Corollary 2.6. *Let A be a separable continuous field of nuclear C^* -algebras over a locally compact metrizable space X that has no compact open subsets. Then A is connective.*

Proof. Let $Y = X \cup \{y_0\}$ be the one-point compactification of X . Then Y is a compact metrizable space which must be connected. Indeed, arguing by contradiction, say that $Y = U \cup V$ with U, V open and nonempty with $y_0 \in U$ and $U \cap V = \emptyset$. Then $V = V \cap X$ is both an open and compact subset of X .

We can view A as a continuous field over Y (see the remark on page 145 of [2]) and the fiber over y_0 satisfies $A(y_0) = \{0\}$. It follows that A is connective by Thm. 2.5. \square

2.4. Obstructions to connectivity. If A is a C^* -algebra, we denote by \widehat{A} the spectrum of A , which consists of all unitary equivalence classes of irreducible representations and by $\text{Prim}(A)$ the primitive spectrum of A consisting of kernels of irreducible representations. The unitary dual \widehat{G} of a group G identifies with $\widehat{C^*(G)}$. Recall that \widehat{A} is topologized by pulling-back the Jacobson topology of $\text{Prim}(A)$ under the natural map $\widehat{A} \rightarrow \text{Prim}(A)$, $\pi \mapsto \ker \pi$, [14]. Let π and $(\pi_n)_n$ be irreducible representations of A acting on the same separable Hilbert space H . Suppose that $\|\pi_n(a)\xi - \pi(a)\xi\| \rightarrow 0$ for all $a \in A$ and $\xi \in H$. Then the sequence $(\pi_n)_n$ converges to π in the topology of \widehat{A} , see [14, Sec. 3.5].

Proposition 2.7. *Let A be a separable C^* -algebra.*

- (i) *If $\text{Prim}(A)$ contains a non-empty compact open subset, then A is not connective.*
- (ii) *If A is nuclear and $\text{Prim}(A)$ is Hausdorff, then A is connective if and only if $\text{Prim}(A)$ does not contain a non-empty compact open subset.*

Proof. (i) Set $X = \text{Prim}(A)$. If X has a non-empty compact open subset, then $A \otimes \mathcal{O}_2$ contains a nonzero projection by [30, Prop. 2.7] and hence A cannot be connective.

(ii) One implication follows from (i). For the other implication suppose that $X = \text{Prim}(A)$ does not contain a non-empty compact open subset. Since X is Hausdorff by assumption, A is a nuclear separable continuous field over the locally compact space X , [16]. This is explained in detail in [3, Sec. 2.2.2]. Now we apply Cor. 2.6. \square

We would like thank Gabor Szabo for pointing out the following invariance property of connectivity.

Proposition 2.8. *Let A and B be separable nuclear C^* -algebra with homeomorphic primitive spectra. Then A is connective if and only if B is connective.*

Proof. Kirchberg's classification theorem [26] implies that if A and B are as in the statement, then $A \otimes \mathcal{O}_2 \otimes \mathcal{K} \cong B \otimes \mathcal{O}_2 \otimes \mathcal{K}$. The desired conclusion follows now from Proposition 2.3. \square

Definition 2.9. Let A be a separable C^* -algebra. A point $\pi \in \widehat{A}$ is called *shielded*, if $\widehat{A} \setminus \{\pi\} \neq \emptyset$ and any sequence $(\pi_n)_n$ in $\widehat{A} \setminus \{\pi\}$ which converges to π also converges to another point $\eta \in \widehat{A} \setminus \{\pi\}$.

Lemma 2.10. *Let A be a unital separable C^* -algebra. If a point $\pi \in \widehat{A}$ is closed and shielded, then $I = \ker \pi$ is not connective.*

Proof. Observe that $I \neq \{0\}$, since $\widehat{A} \setminus \{\pi\} \neq \emptyset$ and $\{\pi\}$ is closed. By Proposition 2.7 it suffices to show that $\text{Prim}(I) = \text{Prim}(A) \setminus \{I\}$ is a nonempty compact-open subset of $\text{Prim}(A)$. Since $\{\pi\}$ is closed, it follows that $q^{-1}(I) = \{\pi\}$ and hence $q(\widehat{A} \setminus \{\pi\}) = \text{Prim}(A) \setminus \{I\}$. The quotient map $q: \widehat{A} \rightarrow \text{Prim}(A)$ is continuous and open, since the topology of \widehat{A} is defined as the preimage of the topology of $\text{Prim}(A)$. Therefore, the lemma follows if we show that $\widehat{A} \setminus \{\pi\}$ is compact and open. $\widehat{A} \setminus \{\pi\}$ is open because $\{\pi\}$ is closed. Since \widehat{A} is compact and satisfies the second axiom of countability [14], it suffices to show that $\widehat{A} \setminus \{\pi\}$ is sequentially compact, [25, p. 138]. Let $(\pi_n)_n$ be a sequence in $\widehat{A} \setminus \{\pi\}$. By compactness of \widehat{A} it contains a subsequence $(\pi_{n_k})_k$ converging in \widehat{A} . If it converges to $\pi \in \widehat{A}$, then it also converges to some other point $\eta \in \widehat{A} \setminus \{\pi\}$, because π is shielded. Hence $\widehat{A} \setminus \{\pi\}$ is also compact. \square

Corollary 2.11. *Let G be a countable discrete group. If the trivial representation $\iota \in \widehat{G}$ is shielded, then $I(G)$ is not connective.*

Proof. Since ι is a one-dimensional representation, it follows that $\{\iota\}$ is closed in \widehat{G} . Thus, the statement follows from Lemma 2.10. \square

3. CONNECTIVITY OF CRYSTALLOGRAPHIC GROUPS

It is known that there are precisely 10 closed flat 3-dimensional manifolds. Conway and Rossetti [7] call these manifolds platycosms (“flat universes”). The Hantzsche-Wendt manifold [17], or the didicosm in the terminology of [7], is the only platycosm with finite homology. Its fundamental group G , known as the Hantzsche-Wendt group, is generated by two elements x and y subject to two relations:

$$x^2yx^2 = y, \quad y^2xy^2 = x.$$

The group G is one of the classic torsion free 3-dimensional crystallographic groups, [17, 7]. It is useful to introduce the notation $z = (xy)^{-1}$.

A concrete realization of G as rigid motions of \mathbb{R}^3 is given by the following transformations X, Y, Z that correspond to the group elements x, y and z .

$$X(\xi) = A\xi + a, \quad Y(\xi) = B\xi + b, \quad Z(\xi) = C\xi + c, \quad \xi \in \mathbb{R}^3,$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$a = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}, \quad c = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix}.$$

The transformations X^2 , Y^2 and Z^2 are just translations by unit vectors in the positive directions of the coordinate axes.

One shows (independently of the previous concrete realization) that the elements x^2 , y^2 and z^2 commute. Moreover one has the following relations in G :

$$\begin{aligned} xx^2x^{-1} &= x^2, & xy^2x^{-1} &= y^{-2}, & xz^2x^{-1} &= z^{-2} \\ yx^2y^{-1} &= x^{-2}, & yy^2y^{-1} &= y^2, & yz^2y^{-1} &= z^{-2} \\ zx^2z^{-1} &= x^{-2}, & zy^2z^{-1} &= y^{-2}, & zz^2z^{-1} &= z^2 \end{aligned}$$

The subgroup N of G generated by x^2 , y^2 and z^2 is normal in G and it is isomorphic to $\mathbb{Z}^3 \cong \mathbb{Z}x^2 \oplus \mathbb{Z}y^2 \oplus \mathbb{Z}z^2$. Let $q : G \rightarrow H = G/N$ denote the quotient map.

$$1 \longrightarrow N \longrightarrow G \xrightarrow{q} H \longrightarrow 1$$

H is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ with generators are $q(x)$ and $q(y)$.

For later use, we will need the following identities that hold in G .

$$(2) \quad \begin{aligned} x^{-1}y &= yxy^2z^{-2} = zx^2z^{-2}, & x^{-1}z &= yz^2, \\ y^{-1}x &= zx^2, & y^{-1}z &= x(x^{-2}y^2). \end{aligned}$$

3.1. Induced representations and the unitary dual of G . Based on Corollary 2.11 we will show that $I(G)$ for the Hantzsche-Wendt group G is not connective. This requires a thorough analysis of the spectrum \widehat{G} .

Our basic reference for this section is the book of Kaniuth and Taylor [23]. The unitary dual of G consists of unitary equivalence classes of irreducible unitary representations of G and is denoted by \widehat{G} . G acts on $\widehat{N} \cong \mathbb{T}^3$ by $g \cdot \chi = \chi(g \cdot g^{-1})$. If we identify the character $\chi \in \widehat{N}$ with the point $(\chi(x^2), \chi(y^2), \chi(z^2)) = (u, v, w) \in \mathbb{T}^3$, then the action of G is described as follows:

$$x \cdot (u, v, w) = (u, \bar{v}, \bar{w}), \quad y \cdot (u, v, w) = (\bar{u}, v, \bar{w}), \quad z \cdot (u, v, w) = (\bar{u}, \bar{v}, w).$$

The stabilizer of a character χ is the subgroup G_χ of G defined by $G_\chi = \{g \in G : \chi(g \cdot g^{-1}) = \chi(\cdot)\}$. It is clear that $N \subset G_\chi$ and that there is a bijection from G/G_χ onto the orbit of χ . In particular, the orbits of the action of G on \widehat{N} can only have length 1, 2 or 4. Mackey has shown that each irreducible representation $\pi \in \widehat{G}$ is supported by the orbit of some character $\chi \in \widehat{N}$, in the sense that the restriction of π to N is unitarily equivalent to some

multiple m_π of the direct sum of the characters in the orbit of χ .

$$\pi|_N \sim m_\pi \bigoplus_{g \in G/G_\chi} \chi(g \cdot g^{-1}).$$

In the sum above g runs through a set of coset representatives.

Mackey's theory has a particularly nice form for virtually abelian discrete groups. Let $\Omega \subset \widehat{N}$ be a subset which intersects each orbit of G exactly once. For each $\chi \in \widehat{N}$, let \widehat{G}_χ be the unitary dual of the stabilizer group G_χ and denote by $\widehat{G}_\chi^{(\chi)}$ the subset of \widehat{G}_χ consisting of classes of irreducible representations σ of G_χ such the restriction of σ to N is unitarily equivalent to a multiple of χ . Then, according to [23, Thm. 4.28]

Theorem 3.1. $\widehat{G} = \left\{ \text{ind}_{G_\chi}^G(\sigma) : \sigma \in \widehat{G}_\chi^{(\chi)}, \chi \in \Omega \right\}.$

Let ι be the trivial representation of G . We will prove that $\iota \in \widehat{G}$ is shielded by showing that any sequence $(\pi_n)_n$ of points in $\widehat{G} \setminus \{\iota\}$ that converges to ι has a subsequence which is convergent to a point $\eta \neq \iota$.

Let $R_\ell \subset \widehat{G}$ consist of those classes of irreducible representations which lie over ℓ -orbits, i.e. the orbits of length ℓ . Write \widehat{G} as the disjoint union $\widehat{G} = R_1 \cup R_2 \cup R_4$. It suffices to assume that all the elements π_n belong to the same subset R_ℓ . We distinguish the three possible cases for ℓ :

1-orbits. Consider the characters of N of the form $\chi = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$, where $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{\pm 1\}$. These are precisely the points in \widehat{N} which are fixed under the action of G . In other words $G_\chi = G$. Let $(\pi_n)_n$ be a sequence of elements in $R_1 \subset \widehat{G}$ and such that $(\pi_n)_n$ is convergent to ι . Since the restriction of π_n to N is a multiple of a character $\chi_n = (\varepsilon_1(n), \varepsilon_2(n), \varepsilon_3(n))$, it follows that $\varepsilon_1(n) = \varepsilon_2(n) = \varepsilon_3(n) = 1$ for all sufficiently large n and hence since π_n is irreducible, there is m such that $\pi_n = \iota$ for $n \geq m$. Hence there is no sequence in $R_1 \setminus \{\iota\}$ which converges to ι .

2-orbits. The characters $\chi \in \widehat{N}$ with orbits of length two are those $\chi = (u, v, w)$ where precisely only one of the coordinates is not equal to ± 1 . Let us argue first that if $(\pi_n)_n$ is a sequence of elements in $R_2 \subset \widehat{G}$ such that π_n lies over the orbit of $\chi_n = (u_n, v_n, w_n)$ and $(\pi_n)_n$ is convergent to ι , then two of the coordinates u_n, v_n, w_n must be equal to 1 for all sufficiently large n .

Suppose that each π_n lies over the orbit of a character χ_n of the form $\chi_n = (u_n, \varepsilon_2(n), \varepsilon_3(n))$ where $u_n \neq \pm 1$ and $\varepsilon_2(n), \varepsilon_3(n) \in \{\pm 1\}$. Then G_{χ_n} is generated by x, y^2, z^2 and $\{e, y\}$ are coset representatives for G/G_{χ_n} .

Since $\pi_n|_N \sim m_n(\chi_n(\cdot) \oplus \chi_n(y \cdot y^{-1}))$, it follows that

$$\pi_n(y^2) \sim m_n \begin{pmatrix} \varepsilon_2(n) & 0 \\ 0 & \varepsilon_2(n) \end{pmatrix}, \quad \pi_n(z^2) \sim m_n \begin{pmatrix} \varepsilon_3(n) & 0 \\ 0 & \varepsilon_3(n) \end{pmatrix}$$

and hence if $(\pi_n)_n$ converges to ι , then we must have $\varepsilon_2(n) = \varepsilon_3(n) = 1$ for all sufficiently large n . The cases $\chi_n = (\varepsilon_1, v_n, \varepsilon_3)$ and $\chi_n = (\varepsilon_1, \varepsilon_2, w_n)$ are treated similarly.

In view of the discussion above, it suffices to focus on characters of N the form $\chi = (u, 1, 1)$. The orbit of χ consists of two points, $(u, 1, 1)$ and $(\bar{u}, 1, 1)$. The corresponding stabilizer G_χ is generated by x, y^2 and z^2 . In particular $G_\chi = N \cup xN$ and $G = G_\chi \cup yG_\chi$. The exact sequence

$$1 \longrightarrow N \longrightarrow G_\chi \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

does not split since G is torsion free. The quotient G/G_χ is generated by the coset of y . Let $\sigma \in \widehat{G}_\chi$ be an irreducible representation of G_χ whose restriction to N is a multiple of χ . Since $\chi(y^2) = \chi(z^2) = 1$, it follows that σ factors through G_χ/\mathbb{Z}^2 . Moreover we have a nontrivial central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow G_\chi/\mathbb{Z}^2 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

where the normal subgroup is generated by the image of x^2 under the map $N \rightarrow N/\langle y^2, z^2 \rangle$ and the quotient group is generated by the image of $q(x)$ under the map $H \rightarrow H/\langle q(y) \rangle$. Since G_χ/\mathbb{Z}^2 is an abelian group, σ must be a character such that $\sigma(x)^2 = \sigma(x^2) = \chi(x^2) = u$. Thus $\sigma(x) = a \in \mathbb{T}$ with $a^2 = u$. Let us compute the representation $\pi = \text{ind}_{G_\chi}^G(\sigma)$ of G induced by σ . It acts on the Hilbert space

$$H_\pi = \{\xi : G \rightarrow \mathbb{C} : \xi(gh) = \sigma(h^{-1})\xi(g), \quad g \in G, h \in G_\chi\}.$$

Since $G = G_\chi \cup yG_\chi$, we can identify H_π with \mathbb{C}^2 via the isometry $\xi \mapsto (\xi(e), \xi(y))$. Then $\pi(g)\xi = \xi(g^{-1}\cdot)$ can be described using (2) as follows:

$$\begin{aligned} \pi(x)\xi(e) &= \xi(x^{-1}) = \sigma(x)\xi(e) = a\xi(e) \\ \pi(x)\xi(y) &= \xi(x^{-1}y) = \xi(yxy^2z^{-2}) = \sigma(z^2y^{-2}x^{-1})\xi(y) = \bar{a}\xi(y) \\ \pi(y)\xi(e) &= \xi(y^{-1}) = \xi(y \cdot y^{-2}) = \sigma(y^2)\xi(y) = \xi(y) \\ \pi(y)\xi(y) &= \xi(y^{-1} \cdot y) = \xi(e) \end{aligned}$$

which produces the following matrices with respect to the basis given above:

$$(3) \quad \pi(x) = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}, \quad \pi(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \pi(z) = \begin{pmatrix} 0 & a \\ \bar{a} & 0 \end{pmatrix}$$

Corresponding to the characters $(1, v, 1)$ and $(1, 1, w)$ we obtain the irreducible representations, where we use the isometries $\xi \mapsto (\xi(e), \xi(x))$ and $\xi \mapsto (\xi(e), \xi(y))$ respectively:

$$(4) \quad \pi(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \pi(y) = \begin{pmatrix} b & 0 \\ 0 & \bar{b} \end{pmatrix}, \quad \pi(z) = \begin{pmatrix} 0 & \bar{b} \\ b & 0 \end{pmatrix}, \quad b^2 = v,$$

and

$$(5) \quad \pi(x) = \begin{pmatrix} 0 & \bar{c} \\ c & 0 \end{pmatrix}, \quad \pi(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \pi(z) = \begin{pmatrix} c & 0 \\ 0 & \bar{c} \end{pmatrix}, \quad c^2 = w.$$

Let $(\pi_n)_n$ be a sequence in R_2 that converges to ι in \widehat{G} . Arguing by symmetry, we may assume that each π_n is given by the formulas (3) corresponding to a sequence of points $u_n \in \mathbb{T}$ with $u_n \notin \{\pm 1\}$. Since $\pi_n \rightarrow \iota$ it follows from the equation (3) that $u_n \rightarrow 1$. Again from (3) we can compute the limits of the sequences $\pi_n(x)$ and $\pi_n(y)$ in $U(2)$. This gives the representation $\pi: G \rightarrow U(2)$:

$$\pi(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \pi(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

It is clear that π is a representation of G that factors through the left regular representation of $\mathbb{Z}/2$. Decompose π into a direct sum of characters $\pi \sim \iota \oplus \eta$. Then η is not equivalent to ι and $\pi_n \rightarrow \eta$ in \widehat{G} .

4-orbits. Let $\chi = (u, v, w) \in \mathbb{T}^3$ be a character of N with $u, v, w \notin \{\pm 1\}$. Its orbit under the action of G consists of four points and $G_\chi = N$. Let us compute the representation $\pi = \text{ind}_N^G(\chi)$ of G induced by σ . It acts on the Hilbert space $H_\pi = \{\xi : G \rightarrow \mathbb{C} : \xi(gh) = \chi(h^{-1})\xi(g), \quad g \in G, h \in N\}$. Thus one can identify H_π with \mathbb{C}^4 via the isometry $\xi \mapsto (\xi(e), \xi(x), \xi(y), \xi(z))$. Using the identities (2), we verify that $\pi(g)\xi = \xi(g^{-1}\cdot)$ is described as follows:

$$\begin{aligned} \pi(x)\xi(e) &= \xi(x^{-1}) = \xi(x \cdot x^{-2}) = \chi(x^2)\xi(x) = u\xi(x) \\ \pi(x)\xi(x) &= \xi(e) \\ \pi(x)\xi(y) &= \xi(x^{-1}y) = \xi(zx^2z^{-2}) = \chi(z^2x^{-2})\xi(z) = w\bar{u}\xi(z) \\ \pi(x)\xi(z) &= \xi(x^{-1}z) = \xi(yz^2) = \chi(z^{-2})\xi(y) = \bar{w}\xi(y) \\ \pi(y)\xi(e) &= \xi(y^{-1}) = \xi(y \cdot y^{-2}) = \chi(y^2)\xi(y) = v\xi(y) \\ \pi(y)\xi(x) &= \xi(y^{-1}x) = \xi(zx^2) = \chi(x^{-2})\xi(z) = \bar{u}\xi(z) \\ \pi(y)\xi(y) &= \xi(e) \\ \pi(y)\xi(z) &= \xi(y^{-1}z) = \xi(x \cdot x^{-2}y^2) = \chi(y^{-2}x^2)\xi(x) = \bar{v}u\xi(x) \end{aligned}$$

producing the matrices:

$$(6) \quad \pi(x) = \begin{pmatrix} 0 & u & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{u}w \\ 0 & 0 & \bar{w} & 0 \end{pmatrix}, \quad \pi(y) = \begin{pmatrix} 0 & 0 & v & 0 \\ 0 & 0 & 0 & \bar{u} \\ 1 & 0 & 0 & 0 \\ 0 & u\bar{v} & 0 & 0 \end{pmatrix}$$

It follows that

$$(7) \quad \pi(x^2) = \begin{pmatrix} u & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & \bar{u} & 0 \\ 0 & 0 & 0 & \bar{u} \end{pmatrix}, \quad \pi(y^2) = \begin{pmatrix} v & 0 & 0 & 0 \\ 0 & \bar{v} & 0 & 0 \\ 0 & 0 & v & 0 \\ 0 & 0 & 0 & \bar{v} \end{pmatrix}, \quad \pi(z^2) = \begin{pmatrix} w & 0 & 0 & 0 \\ 0 & \bar{w} & 0 & 0 \\ 0 & 0 & \bar{w} & 0 \\ 0 & 0 & 0 & w \end{pmatrix}$$

Let $(\pi_n)_n$ be a sequence in R_4 converging to ι in \widehat{G} . Each π_n is given by the formulas (6) corresponding to a sequence of points $(u_k, v_k, w_k) \in \mathbb{T}^3$ with $u_k, v_k, w_k \notin \{\pm 1\}$. Since $\pi_n \rightarrow \iota$ it follows from equation (7) that $u_k, v_k, w_k \rightarrow 1$. Again from (6) we can compute the limits of the sequences $\pi_n(x)$ and $\pi_n(y)$ in the space of unitary operators $U(4)$. This gives the representation $\pi: G \rightarrow U(4)$:

$$\pi(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \pi(y) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

It is clear that π is a representation of G that factors through $H = \mathbb{Z}/2 \times \mathbb{Z}/2$. Decompose π into a direct sum of characters η_i . Since π is not equivalent to a multiple of the trivial representation, it follows that at least one of these characters is not equivalent to ι . On the other hand $\pi_n \rightarrow \eta_i$ in \widehat{G} for all i . Combining the above analysis with Corollary 2.11 we obtain immediately

Corollary 3.2. *If G is the Hantzsche-Wendt group, then $I(G)$ is not connective.*

It was conjectured in [8] that if G is a torsion free discrete amenable group, then $[[I(G), \mathcal{K}]] \cong KK(I(G), \mathbb{C})$. We argue now that this conjecture fails for the Hantzsche-Wendt group. Indeed this follows from the previous corollary in conjunction with the following lemma.

Lemma 3.3. *Let G be a residually finite torsion free discrete amenable group which admits a classifying space with finitely generated K -homology group $K_1(BG)$. Then $[[I(G), \mathcal{K}]] \cong KK(I(G), \mathbb{C})$ if and only if $I(G)$ is connective.*

Proof. Suppose first that $[[I(G), \mathcal{K}]] \cong KK(I(G), \mathbb{C})$. Since G is amenable and residually finite it follows that $C^*(G)$ is residually finite dimensional. Since G is amenable, G satisfies the Baum-Connes conjecture and $C^*(G)$ satisfies the UCT by results of Higson and Kasparov [22] and Tu [35]. In particular we have a short exact sequence

$$0 \rightarrow \text{Ext}^1(K_1(C^*(G)), \mathbb{Z}) \rightarrow KK(C^*(G), \mathbb{C}) \rightarrow \text{Hom}(K_0(C^*(G)), \mathbb{Z}) \rightarrow 0$$

Let $\pi_n: C^*(G) \rightarrow M_{d(n)}(\mathbb{C})$ be a separating sequence of finite dimensional representations. The restriction of π_n to $I(G)$ will be denoted by σ_n . By [8, Prop. 3.2] $(\pi_n)_* = d(n)\iota_*: K_0(C^*(G)) \rightarrow \mathbb{Z}$ and hence $[\sigma_n] \in \text{Ext}^1(K_1(I(G)), \mathbb{Z}) \subset KK(I(G), \mathbb{C})$ is a torsion element since $K_1(I(G)) \cong$

$K_1(BG)$ is finitely generated. After replacing π_n by a suitable multiple of itself we have arranged that $[\sigma_n] = 0$ in $KK(I(G), \mathbb{C})$ and hence $[[\sigma_n]] = 0$ in $[[I(G), \mathcal{K}]]$. Since the sequence (σ_n) separates the elements of $I(G)$ it follows that $I(G)$ is connective.

The converse is contained in the main result of [10] which shows that if A is a separable nuclear connective C^* -algebra, then $[[A, \mathcal{K}]] \cong KK(A, \mathbb{C})$. \square

3.2. Crystallographic groups with cyclic holonomy. In this section we are going to show that torsion free crystallographic groups with cyclic holonomy are connective. Apart from this we isolate a lemma, which proves that $I(G)$ for a group G which is a finite extension of a connective group always contains a “big” connective ideal. In particular, the lemma also holds for the Hantzsche-Wendt group.

The proof of both results uses some tools from the index theory of C^* -subalgebras. A reference is [39]. Let Γ and G be discrete groups and let H be a finite group. Suppose that they fit into an exact sequence

$$1 \longrightarrow \Gamma \longrightarrow G \xrightarrow{q} H \longrightarrow 1 .$$

Let $E: C^*(G) \rightarrow C^*(\Gamma)$ be the faithful conditional expectation [39, Ex. 1.2.3] given on group elements by

$$E(g) = \begin{cases} g & \text{if } g \in \Gamma \\ 0 & \text{else} \end{cases} .$$

Choose a lift $g_h \in G$ for each $h \in H$. The pairs (g_h^{-1}, g_h) form a quasi-basis in the sense of [39, Def. 1.2.2]. Let $\mathcal{E} = C^*(G)$ considered as a right Hilbert $C^*(\Gamma)$ -module, where the right action is induced by the inclusion $C^*(\Gamma) \rightarrow C^*(G)$ and the inner product is given by $\langle a, b \rangle = E(a^*b)$ [39, Sec. 2.1]. Note that \mathcal{E} is complete [39, Prop. 2.1.5]. The quasi-basis induces an isometric isomorphism of right Hilbert $C^*(\Gamma)$ -modules $u: \mathcal{E} \rightarrow \ell^2(H) \otimes C^*(\Gamma)$ with

$$u(a) = \sum_h \delta_h \otimes E(g_h a)$$

and inverse $u^*: \ell^2(H) \otimes C^*(\Gamma) \rightarrow \mathcal{E}$ with $u^*(\delta_h \otimes b) = g_h^{-1} b$. Let $\mathcal{L}_{C^*(\Gamma)}(\mathcal{E})$ be the bounded adjointable operators on \mathcal{E} and denote by $\mathcal{K}_{C^*(\Gamma)}(\mathcal{E})$ the compact ones. Then we have $\mathcal{L}_{C^*(\Gamma)}(\mathcal{E}) \cong \mathcal{K}_{C^*(\Gamma)}(\mathcal{E}) \cong \mathcal{K}(\ell^2(H)) \otimes C^*(\Gamma)$. The left multiplication of $C^*(G)$ on \mathcal{E} induces a $*$ -homomorphism

$$\psi: C^*(G) \rightarrow \mathcal{K}(\ell^2(H)) \otimes C^*(\Gamma)$$

with matrix entries $\psi_{h',h}(a) = E(g_{h'} a g_h^{-1})$. Suppose we have $a \in C^*(G)$ with $\psi(a) = 0$. Then

$$a = \frac{1}{|H|} \sum_{h,h'} g_{h'}^{-1} E(g_{h'} a g_h^{-1}) g_h = 0 .$$

Hence, ψ is injective.

Lemma 3.4. *Let Γ be a connective group and let H be a finite group. Suppose that the group G fits into a short exact sequence of the form*

$$1 \longrightarrow \Gamma \longrightarrow G \xrightarrow{q} H \longrightarrow 1 .$$

Then $I(G, H) = \ker(I(q): I(G) \rightarrow I(H))$ is connective as well.

Proof. Let $\iota: C^*(\Gamma) \rightarrow \mathbb{C}$ be the trivial representation and let ψ be the injective $*$ -homomorphism constructed above. For all $b \in C^*(\Gamma) \subset C^*(G)$ we have $\psi_{h',h}(b) = \delta_{h',h} g_h b g_h^{-1}$. In particular, ψ embeds the ideal J generated by $I(\Gamma)$ into $\ker(\text{id} \otimes \iota) = \mathcal{K}(\ell^2(H)) \otimes I(\Gamma)$, which is connective. Hence J is connective as well. It is clear that $J \subseteq I(G, H)$. Let $x \in I(G, H)$. By the property of the quasi-basis

$$0 = q(x) = \sum_{h \in H} q(E(x g_h^{-1})) h \quad \Rightarrow \quad q(E(x g_h^{-1})) = 0 \quad \forall h \in H .$$

Since $I(G, H) \cap C^*(\Gamma) = I(\Gamma)$ we obtain that $E(x g_h^{-1}) \in I(\Gamma)$ for all $h \in H$ and therefore $x = \sum_h E(x g_h^{-1}) g_h \in J$, hence $J = I(G, H)$. \square

The proof of the second result uses an induction over the rank of the free abelian subgroup based on the following observation.

Lemma 3.5. *Let $m > 1$ and let Γ and G be countable discrete groups that fit into the following short exact sequence*

$$1 \longrightarrow \Gamma \longrightarrow G \xrightarrow{\pi} \mathbb{Z}/m\mathbb{Z} \longrightarrow 1 .$$

Suppose that Γ is connective and there are group homomorphisms $\varphi: G \rightarrow \mathbb{Z}$ and $q: \mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$, such that $\pi = q \circ \varphi$. Then G is connective as well.

Proof. Let $\psi: C^*(G) \rightarrow \mathcal{K}(\ell^2(H)) \otimes C^*(\Gamma)$ be the injective $*$ -homomorphism constructed above and let $\iota: C^*(\Gamma) \rightarrow \mathbb{C}$ be the trivial representation. Observe that $\rho = (\text{id} \otimes \iota) \circ \psi$ satisfies

$$\rho(g \gamma)_{h',h} = \iota(E(g_{h'} g \gamma g_h^{-1})) = \iota(E(g_{h'} g g_h^{-1}) g_h \gamma g_h^{-1}) = \iota(E(g_{h'} g g_h^{-1}))$$

for all $g \in G$ and $\gamma \in \Gamma$. In particular, ρ factors through the $*$ -homomorphism $C^*(G) \rightarrow C^*(\mathbb{Z}/m\mathbb{Z})$ induced by π . By assumption this decomposes as

$$C^*(G) \xrightarrow{\varphi} C^*(\mathbb{Z}) \xrightarrow{q} C^*(\mathbb{Z}/m\mathbb{Z}) .$$

Altogether we obtain that ρ decomposes into a direct sum of one-dimensional representations, each of which is homotopic through representations to the trivial one. Hence, to show that G is connective, it suffices to construct a path through discrete asymptotic morphisms connecting a faithful morphism with a direct sum of copies of ρ .

Choose a path witnessing the connectivity of Γ , i.e. a discrete asymptotic morphism

$$H_n: C^*(\Gamma) \rightarrow C([0, 1]) \otimes M_n(\mathbb{C})$$

such that for $H_n^{(t)} = \text{ev}_t \circ H_n: C^*(\Gamma) \rightarrow M_n(\mathbb{C})$ we have that $H_n^{(0)}$ is faithful and $H_n^{(1)}$ is a multiple of ι . Then $(\text{id}_{M_m(\mathbb{C})} \otimes H_n) \circ \psi$ has the desired properties. Hence, G is connective. \square

We need the following elementary fact:

Lemma 3.6. *Let $a, b > 1$ be integers and consider the exact sequence*

$$0 \longrightarrow \mathbb{Z}/a\mathbb{Z} \xrightarrow{\cdot b} \mathbb{Z}/ab\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/b\mathbb{Z} \longrightarrow 0$$

with $\pi(x) = x \bmod b$. Any generator of $\mathbb{Z}/b\mathbb{Z}$ lifts to a generator of $\mathbb{Z}/ab\mathbb{Z}$.

Proof. Let $\bar{y} \in \mathbb{Z}/b\mathbb{Z}$ be a generator and let $y \in \{0, \dots, b-1\}$ be a representative. Let p_1, \dots, p_s be the distinct prime factors of ab , such that p_1, \dots, p_r for $r \leq s$ are the ones not dividing y and p_{r+1}, \dots, p_s divide y . Since $\gcd(y, b) = 1$, the primes p_{r+1}, \dots, p_s do not divide b . Let $x = y + p_1 \dots p_r b$. We have for $i \in \{1, \dots, r\}$ and $j \in \{r+1, \dots, s\}$

$$\begin{aligned} x &\equiv y \not\equiv 0 \pmod{p_i}, \\ x &\equiv p_1 \dots p_r b \not\equiv 0 \pmod{p_j}. \end{aligned}$$

Hence $\gcd(x, ab) = 1$ and $x \in \mathbb{Z}/ab\mathbb{Z}$ is a generator with $\pi(x) = \bar{y}$. \square

To start the induction we need the following lemma.

Lemma 3.7. *Let G be a countable torsion free discrete group, which fits into an exact sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow G \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

Then G is isomorphic to \mathbb{Z} , hence in particular connective.

Proof. This can be proven by calculating $H^2(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z})$ for all $\mathbb{Z}/m\mathbb{Z}$ -module structures on \mathbb{Z} , but we give a direct argument here.

Let $x \in G$ be a lift of $1 \in \mathbb{Z}/m\mathbb{Z}$. Then G is generated by x and \mathbb{Z} . Moreover, $x^m \in \mathbb{Z} \cap Z(G)$, where $Z(G)$ denotes the center of G . We have $\text{Aut}(\mathbb{Z}) \cong GL_1(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. If $t \in \mathbb{Z}$ denotes a generator, we therefore can only have $xtx^{-1} = t^{-1}$ or $xt = tx$. Suppose the first is true, then

$$x^m = x x^m x^{-1} = x^{-m} \Rightarrow x^{2m} = e$$

contradicting that G is torsion free. Thus, t and x commute and $x^m = t^n$ for some $n \in \mathbb{Z}$. Without loss of generality we can assume $\gcd(m, n) = 1$. Indeed, if $m = m' \ell$ and $n = n' \ell$ with $\ell > 1$, then $(x^{m'} t^{-n'})^\ell = e$ and therefore $x^{m'} = t^{n'}$ also holds in G . Consider

$$\alpha: G \rightarrow \mathbb{Z} \quad ; \quad x^k t^\ell \mapsto k n + \ell m .$$

This is a well-defined group homomorphism, which is easily seen to be bijective as a consequence of $\gcd(m, n) = 1$. \square

Theorem 3.8. *Let G be a countable, torsion free, discrete group, which fits into an exact sequence of the form*

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow G \xrightarrow{\pi} \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

for some $n, m \in \mathbb{N}$. Then G is connective.

Proof. This will be proven by induction over the rank of the free abelian subgroup. The case $n = 1$ follows from Lemma 3.7.

Observe that $Z(G) \neq \{e\}$. Indeed, let $x \in G$ be a lift of the generator of $\mathbb{Z}/m\mathbb{Z}$. Then G is generated by \mathbb{Z}^n and x . Moreover, $x^m \neq e$ since G is torsion free and $\pi(x^m)$ is trivial, hence $x^m \in \mathbb{Z}^n$. Thus, x^m commutes with \mathbb{Z}^n and x , hence with all elements of G , i.e. $x^m \in Z(G)$.

This implies that the transfer homomorphism $T: G \rightarrow \mathbb{Z}^n$ associated to the finite index subgroup \mathbb{Z}^n is non-trivial. Therefore there exists a surjective group homomorphism $\varphi: G \rightarrow \mathbb{Z}$. Let $q: \mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ be the canonical quotient homomorphism and let $\bar{\varphi} = q \circ \varphi$. Let $H = \ker(\varphi)$. We have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & H & \longrightarrow & \mathbb{Z}/a\mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}^n & \longrightarrow & G & \xrightarrow{\pi} & \mathbb{Z}/m\mathbb{Z} \longrightarrow 0 \\ & & \downarrow & & \downarrow \varphi & & \downarrow \\ 0 & \longrightarrow & b\mathbb{Z} & \longrightarrow & \mathbb{Z} & \xrightarrow{\pi'} & \mathbb{Z}/b\mathbb{Z} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

The value of a is chosen in such a way that $\text{Im}(\pi|_H) \cong \mathbb{Z}/a\mathbb{Z}$ and b satisfies $m = ab$. The homomorphism π' is surjective since π and $\mathbb{Z}/m\mathbb{Z} \rightarrow \mathbb{Z}/b\mathbb{Z}$ are. The vertical arrow on the left hand side is induced by $\varphi|_{\mathbb{Z}^n}$.

Suppose $H \subset \ker(\pi) = \mathbb{Z}^n$. Then $a = 1$, $b = m$ and $\pi = \pi' \circ \varphi$. By Lemma 3.5, G is then connective. So we may assume $a > 1$.

We claim that there is an element $g \in G$ such that $\varphi(g) = 1$ and $\pi(g)$ is a generator of $\mathbb{Z}/m\mathbb{Z}$. This is constructed as follows: If $b = 1$, we can choose $g \in G$, such that $\varphi(g) = 1$ and modify it by an element in H to achieve that $\pi(g)$ becomes a generator. Otherwise, choose $g' \in G$ such that $\varphi(g') = 1$ and note that $\pi'(\varphi(g'))$ is a generator of $\mathbb{Z}/b\mathbb{Z}$ by surjectivity. We can lift $\pi'(\varphi(g'))$ to a generator $x \in \mathbb{Z}/m\mathbb{Z}$ by Lemma 3.6. Note that $\pi(g') - x \in \mathbb{Z}/a\mathbb{Z}$ and lift this difference to an element $h \in H$. Let $g = g' h^{-1}$. Then $\varphi(g) = \varphi(g') = 1$ and $\pi(g) = \pi(g') - \pi(h) = \pi(g') - \pi(g') + x = x$.

Let $G' = \ker(\bar{\varphi}) = \{g \in G \mid \varphi(g) = \ell \cdot m \text{ for } \ell \in \mathbb{Z}\} \supset H$. Hence, the following diagram has exact rows.

$$(8) \quad \begin{array}{ccccccc} 1 & \longrightarrow & G' & \longrightarrow & G & \xrightarrow{\bar{\varphi}} & \mathbb{Z}/m\mathbb{Z} \longrightarrow 0 \\ & & \downarrow \varphi|_{G'} & & \downarrow \varphi & & \downarrow = \\ 1 & \longrightarrow & m\mathbb{Z} & \longrightarrow & \mathbb{Z} & \xrightarrow{q} & \mathbb{Z}/m\mathbb{Z} \longrightarrow 0 \end{array}$$

The group G is generated by \mathbb{Z}^n and the element g constructed above. We have $g^m \in \mathbb{Z}^n \cap Z(G)$ and $\varphi(g^m) = m$. In particular, $g^m \in Z(G')$. Let

$$\psi: H \times \mathbb{Z} \rightarrow G' \quad ; \quad (h, k) \mapsto h \cdot g^{mk}.$$

This is a group homomorphism, since g^m is central and it fits into the commutative diagram with exact upper and lower part

$$\begin{array}{ccccc} & & G' & & \\ & \nearrow & \uparrow \psi & \searrow \varphi|_{G'}/m & \\ 1 \longrightarrow & H & & \mathbb{Z} & \longrightarrow 0 \\ & \searrow i & \downarrow & \nearrow \text{pr} & \\ & & H \times \mathbb{Z} & & \end{array}$$

proving that ψ is in fact an isomorphism. By the upper row in diagram (8) and Lemma 3.5, the connectivity of G follows if $H \times \mathbb{Z}$, hence H , is connective [10, Thm. 4.1]. But H fits into a short exact sequence of the form

$$0 \rightarrow A \rightarrow H \rightarrow \mathbb{Z}/a\mathbb{Z} \rightarrow 0$$

where A is the free abelian kernel of the nonzero homomorphism $\mathbb{Z}^n \rightarrow b\mathbb{Z}$ from above, which has rank $(n - 1)$. This completes the induction step. \square

4. CONNECTIVITY OF LIE GROUP C^* -ALGEBRAS

In this section we determine which linear connected nilpotent Lie groups and which linear connected reductive Lie groups have connective reduced C^* -algebras. Let us recall that nilpotent connected Lie groups are liminary as shown by Dixmier [13] and Kirillov [27] and semisimple connected Lie groups are liminary as shown by Harish-Chandra [18].

4.1. Solvable and nilpotent Lie groups. A locally compact group N is compactly generated if $N = \bigcup_n V^n$ for some compact subset V of N . Every connected locally compact group is automatically compactly generated. The structure of abelian compactly generated locally compact groups is known. If N is such a group, then $N \cong \mathbb{R}^n \times \mathbb{Z}^m \times K$ for integers $n, m \geq 0$ and K a compact group, [12, Thm. 4.4.2].

Proposition 4.1. *If G is a second countable locally compact amenable group (for example a solvable Lie group) whose center contains a noncompact closed connected subgroup, then $C^*(G)$ is connective.*

Proof. Let N be a noncompact closed connected subgroup of $Z(G)$. Then, by the structure theorem quoted above, N must have a closed subgroup isomorphic to \mathbb{R} . Consider the central extension:

$$0 \rightarrow \mathbb{R} \rightarrow G \rightarrow H \rightarrow 0.$$

Since G is amenable, by [29, Thm. 1.2] (as explained in [15, Lemma 6.3]), $C^*(G)$ has the structure of a continuous field of C^* -algebras over $\widehat{\mathbb{R}} \cong \mathbb{R}$. The desired conclusion follows from Cor. 2.6 since \mathbb{R} has no compact open subsets. \square

Example 4.2. We give here two examples that complement Proposition 4.1.

- (i) Simply connected solvable Lie groups can have discrete noncompact centers. This is the case for $G = \mathbb{C} \rtimes_{\alpha} \mathbb{R}$ where $\alpha : \mathbb{R} \rightarrow \text{Aut}(\mathbb{C})$ is defined by $\alpha(t)(z) = e^{it}z$ for $t \in \mathbb{R}$ and $z \in \mathbb{C}$. In this case $Z(G) = \{0\} \times 2\pi\mathbb{Z}$.

Nevertheless in this case $C^*(G)$ is connective. Consider the extension

$$0 \rightarrow Z(G) \rightarrow G \rightarrow G/Z(G) \cong \mathbb{C} \times \mathbb{T} \rightarrow 0.$$

Then $C^*(G)$ is a continuous $C(\mathbb{T})$ -algebra whose fiber at 1 is the algebra $C^*(\mathbb{C} \times \mathbb{T})$. Since $C^*(\mathbb{C} \times \mathbb{T}) \cong C_0(\mathbb{R}^2) \otimes c_0(\mathbb{Z})$ is connective, so is $C^*(G)$.

- (ii) Both the real and the complex “ $ax + b$ ” groups

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in F^{\times}, b \in F \right\}$$

where $F = \mathbb{R}$ or $F = \mathbb{C}$ are solvable with trivial center and their C^* -algebras contain a copy of the compacts \mathcal{K} , see [33], and so they are not connective.

Theorem 4.3. *Let G be a (real or complex) linear connected nilpotent Lie group. Then $C^*(G)$ is connective if and only if G is not compact.*

Proof. We view G as a real Lie group. By [37, Chap. 2, Thm. 7.3], if G is a linear connected nilpotent Lie group, then G decomposes as a direct product $G = T \times N$ of a torus T and a simply connected nilpotent group N . If G is compact, then $G = T$ and $C^*(G)$ is isomorphic to a direct sum of \mathbb{C} so that it is not connective. If G is noncompact, then N is nontrivial and so the center of G is given by $Z(G) = T \times Z(N)$, where the center $Z(N)$ of N is isomorphic to \mathbb{R}^n for some $n \geq 1$. We conclude the proof by applying Proposition 4.1. \square

Remark 4.4. It is not true that a liminary (CCR) C^* -algebra is connective if and only if does not have nonzero projections. Indeed

$$A = \left\{ f \in C([0, 1], M_2(\mathbb{C})) : f(0) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, f(1) = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}, \lambda \in \mathbb{C} \right\}$$

does not contain nonzero projections but is not connective since $\text{Prim}(A)$ is homeomorphic to a circle S^1 and hence it is compact (and open in itself).

4.2. Reductive Lie groups. A linear connected reductive group G is a closed group of real or complex matrices that is closed under conjugate transpose. In other words G is a closed and selfadjoint subgroup of the general linear group over either \mathbb{R} or \mathbb{C} . A linear connected semisimple group is a linear connected reductive group with finite center [28].

Say $G \subset GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$. Define $K = G \cap O(n)$, or $K = G \cap U(n)$ in the complex case. If G is linear connected reductive, then K is compact, connected and is a maximal compact subgroup of G [28, Prop.1.2].

Let $G = KAN$ be the Iwasawa decomposition of the linear connected semisimple Lie group G . A is abelian and N is nilpotent and both are closed simply connected subgroups of G , [28, Thm. 5.12].

First we consider the case of complex Lie groups.

Theorem 4.5. *If G is a linear connected complex semisimple Lie group, then $C_r^*(G)$ is connective if and only if G is not compact.*

Proof. If G is compact, $C_r^*(G)$ is isomorphic to a direct sum of matrix algebras and hence it is not connective as it contains nonzero projections.

Conversely, suppose now that G is a non-compact linear connected semisimple complex Lie group. Note that from the Cartan decomposition $G =$

KAK [28, Thm. 5.20] it follows that since G is non-compact, so is A and therefore $A \cong \mathbb{R}^n$, for some $n \geq 1$.

Let M be the centralizer of A in K . By Lemma 3.3 and Proposition 4.1 of [31], it follows that

$$C_r^*(G) \subset C_0(\widehat{M} \times \widehat{A}, \mathcal{K}).$$

Since $\widehat{M} \times \widehat{A} \cong \widehat{M} \times \mathbb{R}^n$ does not have nonempty compact open subsets, it follows from Proposition 2.7(ii) that $C_0(\widehat{M} \times \widehat{A}, \mathcal{K})$ is connective. This completes the proof since connectivity passes to C^* -subalgebras. \square

Next we consider the case of linear connected real reductive Lie groups. An element $g \in G$ is semisimple if it can be diagonalized over \mathbb{C} when viewed as a matrix $g \in M_n(\mathbb{C})$.

A closed subgroup H of G is called a Cartan subgroup if it is a maximal abelian subgroup consisting of semisimple elements, [21, p.67]. If G is either compact or a complex Lie group, then all Cartan subgroups of G are connected and they are conjugated inside G . In the general case G has finitely many Cartan subgroups up to conjugacy and Cartan subgroups can have finitely many connected components.

We denote by $\widehat{G}_d \subset \widehat{G}$ the discrete series representations. It consists of unitary equivalence classes of square-integrable representations

$$\sigma : G \rightarrow U(H_\sigma).$$

Harish-Chandra has shown that the discrete series representations of a semisimple Lie group G are parametrized by compact Cartan subgroups and in particular G has discrete series representations if and only if it has a compact Cartan subgroup, [19, 20].

We recall the following facts from [21, p.72] concerning cuspidal parabolic subgroups. Let H be a Cartan subgroup of G . Then H decomposes as a direct product $H = TA = T \times A$, where T is an abelian compact group and A is a vector group isomorphic to \mathbb{R}^n for $n \geq 0$. The case $n = 0$ occurs when H is a compact Cartan subgroup. The centralizer of A in G denoted by

$$L = C_G(A) = \{g \in G : ga = ag, \forall a \in A\}$$

is a Levi subgroup of G . This means that there is a parabolic subgroup of G of the form $P = LN$ (not unique) with L as Levi subgroup. Since A is central in L , H is a relatively compact Cartan subgroup of L , i.e. $H/Z(L)$ is compact. This implies that L has discrete series representations. Such a parabolic subgroup $P = LN$ is called cuspidal.

One can further decompose $L = MA$ to obtain a Langlands decomposition

$$P = MAN = MA \ltimes N,$$

with N a unipotent group. If H is a compact Cartan subgroup, then $L = P = G$ by [21, p.72].

We will write $P = M_P A_P N_P$ whenever we want emphasize the components of P .

The description of $C_r^*(G)$ relies on the analysis of the unitary principal series representations of G associated to parabolic cuspidal subgroups P (also called the P -principal series). They are of the form

$$\text{Ind}_P^G(\sigma \otimes \omega \otimes 1_N),$$

where $\sigma \in \widehat{M_d}$ and $\omega \in \widehat{A}$ and 1_N is the trivial representation of N .

Consider two pairs (P_i, σ_i) , $i = 1, 2$ consisting of cuspidal parabolic subgroups of G and irreducible square-integrable unitary representations of the subgroups M_i , $i = 1, 2$. We say that the pairs are *associated* if there is $g \in G$ such that $gP_1g^{-1} = P_2$ and $\sigma_1(g \cdot g^{-1})$ is unitarily equivalent to σ_2 . This is an equivalence relation [5, Def. 5.2]. We denote by $[P, \sigma]$ the equivalence class of the pair (P, σ) .

The following statement is based on the calculation of $C_r^*(G)$ by A. Wassermann [38] although we don't really use the full strength of his results. An expanded treatment of the structure of $C_r^*(G)$ appears in [5].

Let $J(G) = \bigcap_{\pi \in \widehat{G_d}} \ker(\pi) \subset C_r^*(G)$ be the common kernel of the discrete series representations. The following theorem shows that the K-homology of $C_r^*(G)$ can be described in terms of homotopy classes asymptotic morphisms $C_r^*(G) \rightarrow \mathcal{K}$ which factor through $J(G)$ and discrete series representations.

Theorem 4.6. *Let G be a linear connected real reductive Lie group. Then $C_r^*(G) \cong J(G) \oplus \bigoplus_{\sigma \in \widehat{G_d}} K(H_\sigma)$ and $J(G)$ is a connective liminary C^* -algebra. Moreover, the following assertions are equivalent:*

- (i) $C_r^*(G)$ is connective,
- (ii) G does not have discrete series representations,
- (iii) G does not have a compact Cartan subgroup,
- (iv) There are no nonzero projections in $C_r^*(G)$.

Proof. As explained in [38, p.560], [5, p.1306] the reduced C^* -algebra of a linear reductive connected Lie group admits an embedding

$$C_r^*(G) \hookrightarrow \bigoplus_{[P, \sigma]} C_0(\widehat{A}_P, \mathcal{K}(H_\sigma)),$$

where the direct sum is over equivalence classes $[P, \sigma]$ as above. It is important to emphasize that if G has a compact Cartan subgroup, then G itself is one of the cuspidal parabolic subgroups and we have:

$$C_r^*(G) \hookrightarrow \bigoplus_{[P, \sigma]} C_0(\widehat{A}_P, \mathcal{K}(H_\sigma)) \oplus \bigoplus_{\sigma \in \widehat{G_d}} \mathcal{K}(H_\sigma),$$

where the first direct sum involves proper cuspidal parabolic subgroups $P = M_P A_P N_P$ and hence $\dim(\widehat{A}_P) > 0$. Moreover by [38], [5]:

$$(9) \quad C_r^*(G) \cong J(G) \oplus \bigoplus_{\sigma \in \widehat{G}_d} \mathcal{K}(H_\sigma)$$

where

$$J(G) \hookrightarrow \bigoplus_{[P, \sigma]} C_0(\widehat{A}_P, \mathcal{K}(H_\sigma)) .$$

Hence, $J(G)$ is connective being a subalgebra of a connective C^* -algebra. The first part of the statement follows now from the decomposition (9).

The equivalence (ii) \Leftrightarrow (iii) is Harish-Chandra's result mentioned earlier. In view of the decomposition (9), (ii) implies that $C_r^*(G) = J(G)$ and hence (i) since $J(G)$ is always connective. Connective C^* -algebras do not contain nonzero projections and hence (i) \Rightarrow (iv). Finally by using (9) again, we see that (iv) \Rightarrow (ii) since $\mathcal{K}(H_\sigma)$ contains nonzero projections if $H_\sigma \neq 0$. \square

4.3. A remark on full C^* -algebras of Lie groups. The full C^* -algebra $C^*(G)$ of a property (T) Lie group G contains nonzero projections and hence it is not connective, see [36]. Nevertheless, inspection of several classes of examples indicates that $C^*(G)$ has interesting connective ideals that arise naturally from the representation theory of G . We postpone a detailed discussion of what is known for another time, but would like to mention two examples.

If G is a connected semisimple Lie group with finite center, then $C^*(G)$ is liminary (or CCR), see [40, p.115].

Proposition 4.7. (a) $C^*(SL_2(\mathbb{C}))$ is connective.

(b) $C^*(SL_3(\mathbb{C})) = I(SL_3(\mathbb{C})) \oplus \mathbb{C}$ and $I(SL_3(\mathbb{C}))$ is connective.

Proof. (a) $C^*(SL_2(\mathbb{C}))$ was computed by Fell [16, Thm. 5.4]. We describe now his result. Let Z be the subspace of \mathbb{R}^2 defined by $Z = \bigcup_{n=0}^{\infty} \{n\} \times L_n$ where $L_0 = (-1, \infty)$ and $L_n = (-\infty, \infty)$ for all $n \geq 1$. Endow Z with the induced topology from \mathbb{R}^2 . Let H_0 be a separable infinite dimensional Hilbert space, let $H = H_0 \oplus \mathbb{C}$ and fix a unitary operator $V : H_0 \rightarrow H$. Then $C^*(SL_2(\mathbb{C}))$ is isomorphic to

$$\{F \in C_0(Z, \mathcal{K}(H)) : F(0, -1) = V^* F(2, 0) V \oplus \lambda, \text{ for some } \lambda \in \mathbb{C}\}$$

Since Z has no nonempty open compact subsets it follows that $C_0(Z, \mathcal{K}(H))$ is connective and therefore so is its subalgebra $C^*(SL_2(\mathbb{C}))$.

(b) This will be obtained as a consequence of the following result on the structure of $C^*(SL_3(\mathbb{C}))$ obtained by Pierrot [32]. Let $G = SL_3(\mathbb{C})$ and denote by $\lambda_G : C^*(G) \rightarrow C_r^*(G)$ the morphism induced by the left regular representation and by $\iota_G : C^*(G) \rightarrow \mathbb{C}$ the trivial representation. Pierrot

proved that the kernel J of the morphism $\lambda_G \oplus \iota_G : C^*(G) \rightarrow C_r^*(G) \oplus \mathbb{C}$ is a contractible C^* -algebra. The representation ι_G is isolated since G has property (T). Therefore there is an exact sequence

$$0 \rightarrow J \rightarrow I(G) \rightarrow C_r^*(G) \rightarrow 0$$

where J is contractible and $C_r^*(G)$ is connective by Theorem 4.5. We conclude that $I(G)$ is connective by applying Theorem 2.4. \square

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